On the scalar sector of the covariant graviton two-point function in de Sitter spacetime

Atsushi Higuchi[1] and Spyros S. Kouris[2]

Department of Mathematics, University of York

Heslington, York, YO10 5DD, United Kingdom

[1]Email: ah28@york.ac.uk

[2]Email: ssk101@york.ac.uk

22 May, 200

Abstract

We examine the scalar sector of the covariant graviton two-point function in de Sitter spacetime. This sector consists of the pure-trace part and another part described by a scalar field. We show that it does not contribute to two-point functions of gauge-invariant quantities. We also demonstrate that the long-distance growth present in some gauges is absent in this sector for a wide range of gauge parameters.

1 Introduction

The covariant graviton two-point function in de Sitter spacetime (CGTF) has been studied by several authors [1]–[7]. (See, e.g., [8] for a description of de Sitter spacetime.) It is known that the CGTF grows with the distance between the two points in some gauges [2, 4], although infrared divergences have been shown to be absent [1]. In particular, it is known that the pure-trace part of the CGTF grows with the distance in a gauge where the traceless part is divergence-free [4]. (We call this gauge the Landau gauge in this paper.)

Recently, a non-covariant physical graviton two-point function, which contains only the two physical polarizations, was computed in open de Sitter spacetime [9]. It was

found that this two-point function does not grow as the distance between the two points increases. Subsequently, the present authors showed [10] that the logarithmic growth of the non-covariant physical two-point function in spatially-flat de Sitter spacetime [12] is a gauge artefact. These results suggest that the seemingly problematic long-distance behaviour in the pure-trace part of the CGTF may also be a gauge artefact. Interestingly, the one-loop effect in pure-trace external gravitational fields has been shown to vanish [6]. In fact one of the present authors (AH) showed under some assumptions [11] that the contribution from the pure-trace part and that from another scalar part of the CGTF together take a pure-gauge form for a one-parameter family of gauges which includes the Landau gauge as a limit.[1] (We will call the sector consisting of these two parts the scalar sector.) In this paper we prove this fact without making any assumptions, thus confirming that the scalar sector of the CGTF does not contribute to two-point functions of physical quantities at the tree level. We also emphasize that the mass of the scalar sector is gauge dependent and can be chosen to be a value for which there is no long-distance growth.

The rest of the paper is organized as follows. In section 2 we recall some essential facts about the scalar field theory in de Sitter spacetime as a preliminary. In section 3 we examine the field equations of linearized gravity in a one-parameter family of covariant gauges and identify the scalar sector, which will be the subject of this paper. In section 4 we find some commutator functions, which will be used to find the scalar-sector two-point function. In section 5 we present our main result, i.e. the scalar sector of the CGTF, and make some remarks about the full CGTF. In Appendix A an explicit form of the scalar sector two-point function is presented for a particular value of the gauge parameter, where some simplification occurs.

The metric signature is $(-+++)$ and we set $c = \hbar = 1$ throughout this paper.

2 Scalar field theory

Most of the discussions in this paper are independent of the explicit metric, but for definiteness we work mainly with the metric

$$ds^2 = -dt^2 + H^{-2}\cosh^2 Ht\, dS_3^2\,, \tag{1}$$

[1] Allen and Turyn [2] have also pointed out that the terms which grow at large distances in these two parts of the CGTF are in a pure-gauge form in a particular gauge in the Euclidean approach.

where dS_3^2 is the line element of the unit 3-sphere. The Hubble constant is H and the cosmological constant is $3H^2$ here.

Let us consider the scalar field theory with the Lagrangian density

$$\mathcal{L}_S = -\frac{\sqrt{-g}}{2}\left[\nabla_a\phi\nabla^a\phi + \frac{12H^2}{\alpha-3}\phi^2\right], \tag{2}$$

where ∇_a is the covariant derivative in the background de Sitter spacetime. Indices are raised and lowered by the de Sitter metric g_{ab} given by (1). The mass parameter, $12H^2/(\alpha-3)$, is given in a form which will be useful later.[2] The corresponding Euler-Lagrange equation is

$$L^{(S)}\phi \equiv \left(\Box - \frac{12H^2}{\alpha-3}\right)\phi = 0\,, \tag{3}$$

where $\Box \equiv \nabla_a\nabla^a$. We expand the field ϕ as

$$\phi = \sum_{l\sigma}\left(a_{l\sigma}\phi^{(l\sigma)} + a_{l\sigma}^\dagger\overline{\phi^{(l\sigma)}}\right), \tag{4}$$

where the mode functions $\phi^{(l\sigma)}$ are given by

$$\phi^{(l\sigma)} = f_l(t)Y_{l\sigma} \tag{5}$$

with the $Y_{l\sigma}$ being spherical harmonics on the unit 3-sphere with angular momentum l. They satisfy $\Delta Y_{l\sigma} = -l(l+2)Y_{l\sigma}$, where Δ is the Laplace operator on the unit 3-sphere. (See, e.g., [13] for a concise description of spherical harmonics in higher dimensions.) The label σ distinguishes the spherical harmonics with the same value of l. We require

$$\int_{S^3} d\Omega\,\overline{Y_{l\sigma}}Y_{l'\sigma'} = \delta_{ll'}\delta_{\sigma\sigma'}\,. \tag{6}$$

The functions $f_l(t)$ satisfy

$$\left[\frac{d^2}{dt^2} + 3H\tanh Ht\frac{d}{dt} + \frac{l(l+2)H^2}{\cosh^2 Ht} + \frac{12H^2}{\alpha-3}\right]f_l(t) = 0\,. \tag{7}$$

We normalize the functions $f_l(t)$ by requiring the following Wronskian:

$$W(\overline{f_l}, f_l) \equiv \overline{f_l}\frac{df_l}{dt} - f_l\frac{d\overline{f_l}}{dt} = -\frac{iH^3}{\cosh^3 Ht}\,. \tag{8}$$

Given any two solutions $\phi^{(1)}$ and $\phi^{(2)}$, the following current is conserved:

$$\mathcal{J}^c(\phi^{(1)},\phi^{(2)}) = \phi^{(2)}\nabla^c\phi^{(1)} - \phi^{(1)}\nabla^c\phi^{(2)}\,. \tag{9}$$

[2] The theory may not be well defined for some values of α. We avoid these values.

We define the Klein-Gordon inner product of $\phi^{(1)}$ and $\phi^{(2)}$ as

$$(\phi^{(1)}|\phi^{(2)}) \equiv i\int_\Sigma d\Sigma_c \mathcal{J}^c(\overline{\phi^{(1)}},\phi^{(2)})\,, \tag{10}$$

where Σ is any Cauchy surface and where $d\Sigma_c = d\Sigma\, n_c$ with n_c being the future-pointing unit normal to Σ. (This can be generalized to fields of higher spin. See, e.g. [14].) Here, $d\Sigma = d\theta_1 d\theta_2 d\theta_3\sqrt{\eta}$, where the θ_i are the coordinates on Σ and η is the determinant of the metric on Σ. The product $(\phi^{(1)}|\phi^{(2)})$ is independent of the Cauchy surface Σ because the current $\mathcal{J}^c$ is conserved. We find using (6) and (8)

$$\begin{aligned}
(\phi^{(l\sigma)}|\phi^{(l'\sigma')}) &= \delta_{ll'}\delta_{\sigma\sigma'}\,, && (11)\\
(\overline{\phi^{(l\sigma)}}|\phi^{(l'\sigma')}) &= 0\,. && (12)
\end{aligned}$$

The usual canonical quantization leads to the commutation relations

$$\left[a_{l\sigma}, a^\dagger_{l'\sigma'}\right] = \delta_{ll'}\delta_{\sigma\sigma'} \tag{13}$$

with all other commutators vanishing. We define the vacuum state $|0\rangle$ by requiring $a_{l\sigma}|0\rangle = 0$ for all l and σ. There is some freedom in choosing $f_l(t)$ and, consequently, in choosing the vacuum state. The standard choice leads to the so-called Euclidean [15] (or Bunch-Davies [16]) vacuum, which is invariant under the de Sitter group. We choose this vacuum, but the explicit form of $f_l(t)$ is not necessary here (see, e.g., [17]).

The two-point function of the scalar field ϕ is expressed as

$$\begin{aligned}
\Delta_+(x,x') &\equiv \langle 0|\phi(x)\phi(x')|0\rangle \\
&= \sum_{l\sigma}\phi^{(l\sigma)}(x)\overline{\phi^{(l\sigma)}(x')}\,, && (14)
\end{aligned}$$

where x and x' are spacetime points. The function $\Delta_+(x,x')$ has been calculated by several authors [16, 18, 19]. For example, Allen and Jacobson [19] give it as

$$\Delta_+(x,x') = \frac{\Gamma(a_+)\Gamma(a_-)H^2}{16\pi^2}F(a_+,a_-;2;z)\,, \tag{15}$$

where

$$\begin{aligned}
a_\pm &= \frac{3}{2}\pm\left(\frac{9}{4}-\frac{M^2}{H^2}\right)^{1/2} \quad \text{with} \quad M^2 = \frac{12H^2}{\alpha-3}\,, && (16)\\
z &= \cos^2\left(\frac{\mu(x,x')H}{2}\right)\,. && (17)
\end{aligned}$$

Here, the function $\mu(x,x')$ is the spacelike geodesic distance between points x and x', and $F(a,b;c;z)$ is the hypergeometric function. The variable z can be extended to the

case where there is no spacelike geodesic between x and x'. If we write the de Sitter metric as

$$ds^2 = \frac{1}{H^2\lambda^2}(-d\lambda^2 + d\mathbf{x}^2)\,, \tag{18}$$

with $\mathbf{x} = (x_1, x_2, x_3)$, then for $x = (\lambda, \mathbf{x})$ and $x' = (\lambda', \mathbf{x}')$ we have

$$z = \frac{(\lambda+\lambda')^2 - \|\mathbf{x}-\mathbf{x}'\|^2}{4\lambda\lambda'}\,. \tag{19}$$

(This can readily be inferred by comparing the two-point functions in [16] and [19].) Hence, the large-distance limit corresponds to the limit $z \to -\infty$. The large-distance limit of (15), which will be useful later, can be found as $|\Delta_+(x,x')| \sim (-z)^{-a_-}$ if $0 < M^2 \leq 9/4$ and $|\Delta_+(x,x')| \sim (-z)^{-3/2}$ if $9/4 \leq M^2$, up to a constant factor. Thus, if $M^2 > 0$, i.e. if $\alpha > 3$, the scalar two-point function $\Delta_+(x,x')$ tends to zero as $z \to -\infty$.

The commutator function, which is often called the Schwinger function, is

$$[\phi(x), \phi(x')] = \Delta_+(x,x') - \Delta_+(x',x)\,. \tag{20}$$

Now, define the advanced and retarded Green functions, $G^{(S,A)}(x,x')$ and $G^{(S,R)}(x,x')$, by requiring that

$$L_x^{(S)} G^{(S,A/R)}(x,x') = \delta^4(x,x')\,, \tag{21}$$

and that $G^{(S,A)}(x,x')$ ($G^{(S,R)}(x,x')$) vanish if x is in the future (past) of x'. (The subscript x in $L_x^{(S)}$ indicates that the differential operator $L^{(S)}$ acts at x rather than at x'.) We have defined $\delta^4(x,y)$ by

$$\int dV_y\, \delta^4(x,y) f(y) = f(x) \tag{22}$$

for any smooth and compactly-supported function $f(x)$, where

$$dV_y \equiv d^4y\,\sqrt{-g(y)}\,. \tag{23}$$

Then, as pointed out by Peierls [20], the commutator function is given by the advanced-minus-retarded Green function

$$E^{(S)}(x,x') \equiv G^{(S,A)}(x,x') - G^{(S,R)}(x,x') \tag{24}$$

as

$$[\phi(x), \phi(x')] = iE^{(S)}(x,x')\,. \tag{25}$$

3 The scalar-sector mode functions

The Lagrangian density for linearized gravity can be chosen as

$$\mathcal{L}_{\rm inv} = \sqrt{-g}\left[\frac{1}{2}\nabla_a h^{ac}\nabla^b h_{bc} - \frac{1}{4}\nabla_a h_{bc}\nabla^a h^{bc} + \frac{1}{4}(\nabla^a h - 2\nabla^b h^a{}_b)\nabla_a h \right. \\ \left. -\frac{1}{2}H^2\left(h_{ab}h^{ab} + \frac{1}{2}h^2\right)\right] \tag{26}$$

with $h = h^a{}_a$. This Lagrangian density is invariant under the gauge transformation

$$h_{ab} \to h_{ab} + \nabla_a \Lambda_b + \nabla_b \Lambda_a$$

up to a total divergence. Therefore, one needs to fix the gauge for the canonical quantization of h_{ab}. For this purpose we add the following gauge-fixing term in the Lagrangian density:

$$\mathcal{L}_{\rm gf} = -\frac{\sqrt{-g}}{2\alpha}\left(\nabla_a h^{ab} - \frac{1+\beta}{\beta}\nabla^b h\right)\left(\nabla^c h_{cb} - \frac{1+\beta}{\beta}\nabla_b h\right). \tag{27}$$

Then the Euler-Lagrange field equations derived from $\mathcal{L}_{\rm inv} + \mathcal{L}_{\rm gf}$ are

$$\begin{aligned} L^{(T)cd}_{ab} h_{cd} \equiv & \frac{1}{2}\Box h_{ab} - \left(\frac{1}{2} - \frac{1}{2\alpha}\right)(\nabla_a\nabla_c h^c{}_b + \nabla_b\nabla_c h^c{}_a) \\ & + \left[\frac{1}{2} - \frac{\beta+1}{\alpha\beta}\right]\nabla_a\nabla_b h + \left[\frac{(\beta+1)^2}{\alpha\beta^2} - \frac{1}{2}\right] g_{ab}\Box h \\ & + \frac{1}{2}g_{ab}\left(1 - \frac{2(1+\beta)}{\alpha\beta}\right)\nabla_c\nabla_d h^{cd} - H^2\left(h_{ab} + \frac{1}{2}g_{ab}h\right) = 0\,. \end{aligned} \tag{28}$$

The scalar sector of the field h_{ab} satisfying this equation can be extracted as follows [11]. First, by taking the trace of equation (28) we find

$$\left[\frac{4(1+\beta)^2}{\alpha\beta^2} - \frac{1+\beta}{\alpha\beta} - 1\right]\Box h - 3H^2 h + \left(1 - \frac{3}{\alpha} - \frac{4}{\alpha\beta}\right)\nabla^a\nabla^b h_{ab} = 0\,. \tag{29}$$

There is mixing between the trace h and the traceless part of h_{ab} in general as can be seen from this equation. This mixing can be avoided by choosing the parameter β as

$$\beta = \frac{4}{\alpha - 3}\,. \tag{30}$$

We make this choice in the rest of this paper. We also assume that $\alpha \neq 0$, 3 though we will consider the limit $\alpha \to 0$ in the concluding section. Equation (29) now reads

$$\left(\Box - \frac{12H^2}{\alpha - 3}\right) h = 0\,. \tag{31}$$

We define the traceless part $h_{ab}^{(l)}$ of h_{ab} by

$$h_{ab}^{(l)} \equiv h_{ab} - h_{ab}^{(t)}\,, \tag{32}$$

where

$$h_{ab}^{(t)} \equiv \frac{1}{4}\, g_{ab} h(x)\,. \tag{33}$$

By substituting (32) in (28) with the choice (30) we find

$$\begin{aligned} &\frac{1}{2}\Box h_{ab}^{(l)} - \left(\frac{1}{2} - \frac{1}{2\alpha}\right)\left(\nabla_a \nabla_c h^{(l)c}{}_b + \nabla_b \nabla_c h^{(l)c}{}_a\right) \\ &+ \left(\frac{1}{4} - \frac{1}{4\alpha}\right) g_{ab} \nabla_c \nabla_d h^{(l)cd} - H^2 h_{ab}^{(l)} = 0\,. \end{aligned} \tag{34}$$

Next, define a scalar field B by

$$B \equiv \frac{(\alpha-3)^2}{36\alpha H^4} \nabla^a \nabla^b h_{ab}^{(l)}\,. \tag{35}$$

By using the fact that the Riemann tensor takes the form $R_{abcd} = H^2(g_{ac}g_{bd} - g_{ad}g_{bc})$, we find

$$\left(\Box - \frac{12H^2}{\alpha - 3}\right) B = 0\,. \tag{36}$$

Note that the two fields h and B satisfy the same equation as the scalar field ϕ discussed in the previous section. Hence, we can expand these fields as

$$B(x) = \sum_{l\sigma} \left[b_{l\sigma}\phi^{(l\sigma)}(x) + b_{l\sigma}^{\dagger} \overline{\phi^{(l\sigma)}(x)} \right]\,, \tag{37}$$

$$h(x) = \sum_{l\sigma} \left[c_{l\sigma}\phi^{(l\sigma)}(x) + c_{l\sigma}^{\dagger} \overline{\phi^{(l\sigma)}(x)} \right]\,. \tag{38}$$

Let us write the traceless field $h_{ab}^{(l)}$ as

$$h_{ab}^{(l)} = h_{ab}^{(r)} + h_{ab}^{(d)}\,, \tag{39}$$

where

$$h_{ab}^{(d)} = \left(\nabla_a \nabla_b - \frac{3H^2}{\alpha - 3} g_{ab}\right) B\,. \tag{40}$$

The tensor $h_{ab}^{(d)}$ is traceless because of equation (36). By taking the divergence of this equation twice, we find

$$\nabla^a \nabla^b h_{ab}^{(d)} = \frac{36\alpha H^4}{(\alpha-3)^2} B = \nabla^a \nabla^b h_{ab}^{(l)}\,. \tag{41}$$

As a result we have $\nabla^a \nabla^b h_{ab}^{(r)} = 0$. Thus, the field h_{ab} can be decomposed as

$$h_{ab} = h_{ab}^{(r)} + h_{ab}^{(d)} + h_{ab}^{(t)}\,, \tag{42}$$

where $h_{ab}^{(d)}$ and $h_{ab}^{(t)}$ are given by (40) and (33), respectively. We call the field $h_{ab}^{(d)} + h_{ab}^{(t)}$ the scalar sector of h_{ab} and the field $h_{ab}^{(r)}$ the non-scalar sector. The latter satisfies $\nabla^a \nabla^b h_{ab}^{(r)} = h^{(r)a}{}_a = 0$.

4 Commutator functions of the scalar sector

The contribution of the scalar sector to the two-point function can immediately be inferred once we find its contribution to the commutator (or Schwinger) function. For this reason we calculate the latter in this section.

For any compactly-supported scalar function $f(x)$ we define [21]

$$
\begin{aligned}
(\mathbf{A}f)(x) &\equiv \int dV_y G^{(S,A)}(x,y)f(y)\,, && (43)\\
(\mathbf{R}f)(x) &\equiv \int dV_y G^{(S,R)}(x,y)f(y)\,, && (44)\\
(\mathbf{E}f)(x) &\equiv \int dV_y E^{(S)}(x,y)f(y) \\
&= (\mathbf{A}f)(x) - (\mathbf{R}f)(x)\,. && (45)
\end{aligned}
$$

We define the advanced and retarded Green functions, $G^{(T,A)}_{aba'b'}(x,x')$ and $G^{(T,R)}_{aba'b'}(x,x')$, for the tensor equation (28) by requiring that

$$
L_x^{(T)abcd} G^{(T,A/R)}_{cda'b'}(x,x') = \frac{1}{2}\left(\delta^a_{a'}\delta^b_{b'} + \delta^a_{b'}\delta^b_{a'}\right)\delta^4(x,x')\,, \tag{46}
$$

and that $G^{(T,A)}_{aba'b'}(x,x')$ ($G^{(T,R)}_{aba'b'}(x,x')$) vanish if x is in the future (past) of x'. As in the scalar case we define the advanced-minus-retarded Green function by

$$
E^{(T)}_{aba'b'}(x,x') \equiv G^{(T,A)}_{aba'b'}(x,x') - G^{(T,R)}_{aba'b'}(x,x')\,. \tag{47}
$$

For any compactly-supported smooth tensor $h_{ab}(x)$ we define $(\mathbf{A}h)_{ab}(x)$, $(\mathbf{R}h)_{ab}(x)$ and $(\mathbf{E}h)_{ab}(x)$ in the same way as in the scalar case. The equality shown by Peierls in [20] is quite general and holds in this case as well. Thus,

$$
[h_{ab}(x), h_{a'b'}(x')] = iE^{(T)}_{aba'b'}(x,x')\,. \tag{48}
$$

We will derive the commutator function for the scalar sector starting from this formula.

Let us first examine the pure-trace part. We obtain by a straightforward calculation

$$
L^{(T)abcd}\left(g_{cd}F\right) = \frac{\alpha-3}{4} g^{ab} L^{(S)} F \tag{49}
$$

for any smooth function F. Using this equation with $F = \mathbf{R}f$, we have

$$
\begin{aligned}
&\int dV_x\, G^{(T,R)}_{efab}(y,x) L_x^{(T)abcd}\left[g_{cd}(x)(\mathbf{R}f)(x)\right] \\
&= \frac{\alpha-3}{4}\int dV_x\, G^{(T,R)}_{efab}(y,x) g^{ab}(x) f(x)\,. && (50)
\end{aligned}
$$

We integrate by parts so that the operator $L_x^{(T)cdab}$ is applied on $G^{(T,R)}_{efab}(y,x)$ on the left-hand side.[3] Then we use

$$L_x^{(T)cdab} G^{(T,R)}_{efab}(y,x) = \frac{1}{2}\left(\delta^c_e\delta^d_f + \delta^c_f\delta^d_e\right)\delta^4(y,x)\,, \tag{51}$$

which follows from the equality $G^{(T,A)}_{abcd}(x,y) = G^{(T,R)}_{cdab}(y,x)$. Thus, we find

$$g_{ab}(y)\int dV_x G^{(S,R)}(y,x)f(x) = \frac{\alpha-3}{4}\int dV_x\, G^{(T,R)}_{abcd}(y,x)g^{cd}(x)f(x)\,. \tag{52}$$

Since this holds for any compactly-supported smooth function $f(x)$, we have

$$G^{(T,R)}_{abcd}(y,x)g^{cd}(x) = \frac{4}{\alpha-3}g_{ab}(y)G^{(S,R)}(y,x)\,. \tag{53}$$

The same equality holds for the advanced Green functions $G^{(T,A)}_{abcd}(y,x)$ and $G^{(S,A)}(y,x)$. Therefore

$$E^{(T)}_{abcd}(x,y)g^{cd}(y) = \frac{4}{\alpha-3}g_{ab}(x)E^{(S)}(x,y)\,. \tag{54}$$

By taking the trace of equation (48) and using (54) we find

$$[h(x),h(x')] = i\,\frac{16}{\alpha-3}E^{(S)}(x,x')\,, \tag{55}$$

where $h = g^{ab}h_{ab}$, and

$$\left[h^{(r)}_{ab}(x),h(x')\right] = 0\,, \tag{56}$$

$$[B(x),h(x')] = 0\,, \tag{57}$$

where $h^{(r)}_{ab}$ and B are defined in the previous section.

Next, we derive a relation similar to (54) for the traceless part $h^{(d)}_{ab}$ in the scalar sector. The key equation is

$$L^{(T)abcd}\left(\nabla_c\nabla_d - \frac{1}{4}g_{cd}\Box\right)F = \frac{3-\alpha}{4\alpha}\left(\nabla^a\nabla^b - \frac{1}{4}g^{ab}\Box\right)L^{(S)}F\,, \tag{58}$$

which is valid for any smooth function F. Useful identities in deriving this equation are

$$\Box\nabla_a\nabla_b F = 8H^2(\nabla_a\nabla_b - \frac{1}{4}g_{ab}\Box)F + \nabla_a\nabla_b\Box F \tag{59}$$

and

$$\nabla_a\Box\nabla_b F = 3H^2\nabla_a\nabla_b F + \nabla_a\nabla_b\Box F\,. \tag{60}$$

[3]There will be no boundary terms because the common support of $G^{(T,R)}_{abcd}(y,x)$ and $(\mathbf{R}f)(x)$ with fixed y is compact.

By a procedure similar to that which led to (54) we find from (58)

$$E^{(T)}_{aba'b'}(x,x')\left(\overleftarrow{\nabla}^{a'}\overleftarrow{\nabla}^{b'}-\frac{1}{4}\overleftarrow{\Box}_{x'}\,g^{a'b'}(x')\right)=\frac{4\alpha}{3-\alpha}\left(\nabla_a\nabla_b-\frac{1}{4}g_{ab}(x)\Box_x\right)E^{(S)}(x,x')\,. \tag{61}$$

The derivative operators with primed indices act at point x' here and in the rest of this paper. By using this in (48) we obtain

$$[h_{ab}(x),B(x')]=i\,\frac{3-\alpha}{9H^4}\left[\nabla_a\nabla_b-\frac{1}{4}g_{ab}(x)\right]E^{(S)}(x,x')\,, \tag{62}$$

where $B(x)$ is defined by (36). From this equation we find

$$[B(x),B(x')]=i\,\frac{3-\alpha}{9H^4}E^{(S)}(x,x')\,, \tag{63}$$

$$\left[h^{(r)}_{ab}(x),B(x')\right]=0\,, \tag{64}$$

$$[h(x),B(x')]=0\,. \tag{65}$$

5 Scalar sector of the graviton two-point function

The commutator functions found in the previous section can be used to determine the commutators of annihilation and creation operators in the scalar sector. Since the fields $h(x)$ and $B(x)$ commute with the non-scalar sector $h^{(r)}_{ab}(x)$, the operators $b_{l\sigma}$, $b^\dagger_{l\sigma}$, $c_{l\sigma}$ and $c^\dagger_{l\sigma}$ defined in (37) and (38) commute with $h^{(r)}_{ab}(x)$. Also, equation (65) shows that the operators $b_{l\sigma}$ and $b^\dagger_{l\sigma}$ commute with $c_{l\sigma}$ and $c^\dagger_{l\sigma}$. Thus, the graviton two-point function can be written as

$$\langle 0|h_{ab}(x)h_{a'b'}(x')|0\rangle=\Delta^{(r)}_{aba'b'}(x,x')+\Delta^{(s)}_{aba'b'}(x,x')\,, \tag{66}$$

where

$$\Delta^{(r)}_{aba'b'}(x,x') = \langle 0|h^{(r)}_{ab}(x)h^{(r)}_{a'b'}(x')|0\rangle\,, \tag{67}$$

$$\Delta^{(s)}_{aba'b'}(x,x') = \langle 0|h^{(d)}_{ab}(x)h^{(d)}_{a'b'}(x')|0\rangle+\langle 0|h^{(t)}_{ab}(x)h^{(t)}_{a'b'}(x')|0\rangle\,. \tag{68}$$

The commutators of annihilation and creation operators in the scalar sector can be found from equations (55) and (63) as

$$\left[b_{l'\sigma'},b^\dagger_{l\sigma}\right] = \frac{3-\alpha}{9H^4}\delta_{ll'}\delta_{\sigma\sigma'}\,, \tag{69}$$

$$\left[c_{l'\sigma'},c^\dagger_{l\sigma}\right] = \frac{16}{\alpha-3}\delta_{ll'}\delta_{\sigma\sigma'} \tag{70}$$

with all other commutators vanishing. Now, we require that

$$b_{l\sigma}|0\rangle = c_{l\sigma}|0\rangle = 0 \tag{71}$$

for all l and σ. Using the commutators (69) and (70), the expansion of $B(x)$ and $h(x)$ in (37) and (38) and the condition (71), we find

$$\begin{aligned}
\langle 0|B(x)B(x')|0\rangle &= \frac{3-\alpha}{9H^4}\Delta_+(x,x')\,, &(72)\\
\langle 0|h(x)h(x')|0\rangle &= \frac{16}{\alpha-3}\Delta_+(x,x')\,, &(73)
\end{aligned}$$

where $\Delta_+(x,x')$ is expressed in terms of the scalar mode functions $\phi^{(l\sigma)}$ in (14). From these equations and the expressions of $h_{ab}^{(d)}$ and $h_{ab}^{(t)}$ in terms of B and h, we immediately find the scalar sector of the graviton two-point function as

$$\begin{aligned}
\Delta^{(s)}_{aba'b'}(x,x') &= \frac{3-\alpha}{9H^4}\left(\nabla_a\nabla_b - \frac{3H^2}{\alpha-3}g_{ab}\right)\left(\nabla_{a'}\nabla_{b'} - \frac{3H^2}{\alpha-3}g_{a'b'}\right)\Delta_+(x,x') \\
&\quad + \frac{16}{\alpha-3}\times\frac{1}{16}g_{ab}g_{a'b'}\Delta_+(x,x') \\
&= \left(\frac{3-\alpha}{9H^4}\nabla_a\nabla_b\nabla_{a'}\nabla_{b'} + \frac{1}{3H^2}\nabla_a\nabla_b g_{a'b'} + \frac{1}{3H^2}g_{ab}\nabla_{a'}\nabla_{b'}\right) \\
&\quad \times\Delta_+(x,x') \qquad (74)
\end{aligned}$$

with $g_{ab} = g_{ab}(x)$ and $g_{a'b'} = g_{a'b'}(x')$. As we have seen, the function $\Delta_+(x,x')$ decreases at large distances if $\alpha > 3$. Hence the scalar-sector two-point function $\Delta^{(s)}(x,x')$ decreases at large distances for these values of α. Note also that each term in the scalar sector is pure gauge either at point x or x'.[4] Hence, there will be no contribution from this sector to the two-point function of a gauge-invariant quantity. This is in agreement with the expectation that the extra modes introduced in the theory by gauge fixing should not contribute to physical quantities at the tree level. In Appendix A we present an explicit form of the scalar-sector two-point function for $\alpha = 9$, where some simplification occurs.

Let us comment on the limit $\alpha \to 0$, i.e. the Landau gauge considered in [4]. In this limit the field satisfies

$$\nabla^b h_{ab} = \frac{1}{4}\nabla_a h\,. \tag{75}$$

The two-point function $\Delta^{(s)}(x,x')$ remains in a pure-gauge form in this limit. However, the traceless part $h_{ab}^{(l)}$ defined by (39) satisfies $\nabla^a\nabla^b h_{ab}^{(l)} = 0$. (As a result, some intermediate results such as equation (58) are ill-defined.) Therefore, if one imposed the

[4]Related remarks were made in [11] and [2].

condition (75) from the start without taking the $\alpha \to 0$ limit of the theory with nonzero α, the field would be decomposed as $h_{ab} = h^{(r)}_{ab} + h^{(t)}_{ab}$ with $\nabla^a\nabla^b h^{(r)}_{ab} = 0$. Therefore, the cancellation of physical contributions from the fields $h^{(d)}_{ab}$ and $h^{(t)}_{ab}$ could be overlooked.

We have not calculated the non-scalar sector two-point function, $\Delta^{(r)}_{aba'b'}(x,x')$, which is of course necessary for finding the full two-point function. The full two-point function can most easily be calculated by extending the work of Allen and Turyn [2] in the Euclidean approach. (They specialize to the choice $\alpha = 1$ and $\beta = -2$. Since the squared mass of the modes in the scalar sector for this choice is $-6H^2$, their two-point function grows badly at large distances in the scalar sector.) Our preliminary results with arbitrary values of α and β show that it is impossible to construct a covariant two-point function which does not grow at large distances in this manner. However, the results in [9] and [10] imply that this growth in CGTF is also pure gauge. We are currently investigating if this fact can be verified directly.

Acknowledgement

We thank Chris Fewster for useful discussions concerning the relation between the symplectic product and the advanced-minus-retarded Green function and Bernard Kay for helpful discussions on gauge invariant quantities. We also thank Bruce Allen and Michael Turyn for useful communication.

Appendix A. The scalar sector of the two-point function with $\alpha = 9$

Note that for $\alpha = 9$ we have $12H^2/(\alpha - 3) = 2H^2$. Therefore we have the conformally-coupled massless scalar field. In this case the scalar two-point function (14) is

$$\Delta_+(x,x') = \frac{H^2}{16\pi^2}F(2,1;2;z) = \frac{H^2}{16\pi^2}\frac{1}{1-z}\,. \tag{A1}$$

The two-point function $\Delta^{(s)}_{aba'b'}(x,x')$ can be found from (74) using [19]

$$\nabla_a n_b = H\cot H\mu\,(g_{ab} - n_a n_b)\,, \tag{A2}$$

$$\nabla_a n_{b'} = -\frac{H}{\sin H\mu}(g_{ab'} + n_a n_{b'})\,, \tag{A3}$$

$$\nabla_a g_{bc'} = \frac{H(1-\cos H\mu)}{\sin H\mu}\,H(g_{ab}n_{c'} + g_{ac'}n_b)\,, \tag{A4}$$

where $n_a = \nabla_a \mu(x,x')$ is the tangent vector at point x to the spacelike geodesic joining points x and x' (if there is such a spacelike geodesic). The bi-vector $g_{ab'}$ is the parallel propagator, i.e. for any vector $X^{a'}$ at point x', $g_{ab'}X^{a'}$ is the vector at point x obtained by parallelly transporting $X^{a'}$ along the geodesic. The result is

$$\begin{aligned}\Delta^{(s)}_{aba'b'}(x,x') = T_1(z)\, n_a n_b n_{a'} n_{b'} + T_2(z)\,(g_{a'b'} n_a n_b + g_{ab} n_{a'} n_{b'}) \\ +T_3(z)\,(g_{aa'} n_b n_{b'} + g_{ba'} n_a n_{b'} + g_{ab'} n_b n_{a'} + g_{bb'} n_a n_{a'}) \\ +T_4(z)\,(g_{ab'} g_{ba'} + g_{bb'} g_{aa'}) + T_5(z)\, g_{ab} g_{a'b'}\,,\end{aligned} \tag{A5}$$

where the coefficients $T_i(z)$ are given by

$$T_1(z) = \frac{H^2}{24\pi^2}\left[\frac{4}{z-1} + \frac{24}{(z-1)^2} + \frac{24}{(z-1)^3}\right], \tag{A6}$$

$$T_2(z) = -\frac{H^2}{24\pi^2}\left[\frac{1}{z-1} + \frac{4}{(z-1)^2} + \frac{3}{(z-1)^3}\right], \tag{A7}$$

$$T_3(z) = \frac{H^2}{24\pi^2}\left[\frac{2}{(z-1)^2} + \frac{3}{(z-1)^3}\right], \tag{A8}$$

$$T_4(z) = \frac{H^2}{24\pi^2}\left[\frac{1}{2(z-1)^3}\right], \tag{A9}$$

$$T_5(z) = \frac{H^2}{24\pi^2}\left[\frac{1}{(z-1)^2} + \frac{1}{2(z-1)^3}\right], \tag{A10}$$

where $z = \cos^2\left(\frac{\mu(x,x')H}{2}\right)$. As we have seen in section 2 the variable z tends to $-\infty$ as the coordinate distance $r = \|\mathbf{x} - \mathbf{x}'\|$ in the coordinates for metric (18) tends to infinity. In the present case the coefficients T_i tend to zero like r^{-2}. In the rest of this appendix we will calculate the components of the tangent vector n_a and the parallel propagator $g_{ab'}$ and see explicitly that they are bounded as $r \to \infty$. Thus, we will see that the two-point function in (A5) indeed tends to zero as $r \to \infty$.

Let us define

$$\chi(x,x') \equiv \cos H\mu = \frac{\lambda^2 + \lambda'^2 - r^2}{2\lambda\lambda'}\,. \tag{A11}$$

We extend the function $\mu(x,x')$ with $x = (\lambda, \mathbf{x})$ and $x' = (\lambda', \mathbf{x}')$ in the coordinates for metric (18) to the region with $\chi > 1$ by

$$e^{iH\mu} = \chi + \sqrt{\chi^2 - 1}\,. \tag{A12}$$

We then have

$$n_a = -\frac{i}{H\sqrt{\chi^2-1}}\nabla_a \chi \tag{A13}$$

and

$$\nabla_a \chi = -\frac{Hr}{\lambda'} V_a + H\left(\frac{\lambda}{\lambda'} - \chi\right) t_a \,, \tag{A14}$$

where the vectors V^a and t^a at point $x = (\lambda, \mathbf{x})$ are defined by [12]

$$V^0 = 0 \;, \quad V^i = \frac{H\lambda(x^i - x'^i)}{r} \,, \tag{A15}$$

with "0" referring to the λ-component, and

$$t^a = -H\lambda \left(\frac{\partial}{\partial \lambda}\right)^a \,. \tag{A16}$$

The unit vector V^a is spacelike and the vector t^a is a future-pointing unit normal to the Cauchy surface with $\lambda = $ const, if we let the variable λ decrease towards the future. (We define vectors $V^{a'}$ and $t^{a'}$ at x' in a similar manner. We note that $\lambda^{-1} V_i = -(\lambda')^{-1} V_{i'}$ in components.) Thus, we obtain

$$n_a = \frac{ir}{\lambda' \sqrt{\chi^2 - 1}} V_a - \frac{i}{\sqrt{\chi^2 - 1}} \left(\frac{\lambda}{\lambda'} - \chi\right) t_a \,. \tag{A17}$$

We note that, since χ grows like r^2 as r increases, we have $n_a \to -it_a$ as $r \to \infty$.

The expression for the parallel propagator $g_{ab'}$ can be found from equation (A3). Thus, we have

$$g_{ab'} = \frac{1}{H^2} \nabla_a \nabla_{b'} \chi - \frac{1}{H^2(\chi + 1)} \nabla_a \chi \cdot \nabla_{b'} \chi \,. \tag{A18}$$

Let us define a bi-vector $P_{ab'}$ by $t^a P_{ab'} = t^{b'} P_{ab'} = 0$ and

$$P_{ij'} = \frac{1}{\lambda \lambda' H^2} \delta_{ij'} \,. \tag{A19}$$

Then

$$H^{-2} \nabla_a \nabla_{b'} \chi = P_{ab'} + \frac{r}{\lambda} t_a V_{b'} + \frac{r}{\lambda'} t_{b'} V_a + \left(\chi - \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right) t_a t_{b'} \,. \tag{A20}$$

Hence, the bi-vector $g_{ab'}$ can be written as

$$\begin{aligned} g_{ab'} \;&=\; P_{ab'} + \frac{1}{\chi + 1} \left(\chi - \frac{\lambda'}{\lambda} - \frac{\lambda}{\lambda'} - 1\right) t_a t_{b'} - \frac{r^2}{(\chi + 1)\lambda \lambda'} V_a V_{b'} \\ &\quad + \left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right) \frac{r}{\chi + 1} \left(t_a V_{b'} + t_{b'} V_a\right) \,. \end{aligned} \tag{A21}$$

We find $g_{ab'} \to P_{ab'} + t_a t_{b'} + 2 V_a V_{b'}$ as $r \to \infty$.

References

[1] B. Allen, *Phys. Rev. D* **34**, 3670 (1986).

[2] B. Allen and M. Turyn, *Nucl. Phys.* **B292**, 813 (1987).

[3] E. Floratos, J. Iliopoulos and T.N. Tomaras, *Phys. Lett.* **197B**, 373 (1987).

[4] I. Antoniadis and E. Mottola, *J. Math. Phys.* **32**, 1037 (1991).

[5] N.C. Tsamis and R.P. Woodard, *Commun. Math. Phys.* **162**, 217 (1994).

[6] I. Antoniadis, J. Iliopoulos and T.N. Tomaras, *Nucl. Phys.* **B462**, 437 (1996).

[7] M.V. Takook, *The covariant two-point function for linear gravity in de Sitter space*, gr-qc/0001052.

[8] S.W. Hawking and G.F.R. Ellis, *The large-scale structure of space-time* (Cambridge Univ., Cambridge, 1973).

[9] S.W. Hawking, T. Hertog and N. Turok, *Phys. Rev. D* **62**, 063502 (2000).

[10] A. Higuchi and S.S. Kouris, *Class. Quantum Grav.* **17**, 3077 (2000).

[11] A. Higuchi, *Nucl. Phys.* **B282**, 397 (1987).

[12] B. Allen, *Nucl. Phys.* **B287**, 743 (1987).

[13] A. Chodos and E. Myers, *Ann. Phys. (N.Y.)* **156**, 412 (1984).

[14] J.L. Friedman, *Commun. Math. Phys.* **62**, 247 (1978).

[15] G. Gibbons and S.W. Hawking, *Phys. Rev. D* **15**, 2738 (1977).

[16] T.S. Bunch and P.C.W. Davies, *Proc. R. Soc. London* **A360**, 117 (1978).

[17] E. Tagirov, *Ann. Phys. (N.Y.)*, **76**, 561 (1973).

[18] C. Schomblond and P. Spindel, *Ann. Inst. Henri Poincaré*, **25A**, 67 (1976);

[19] B. Allen and T. Jacobson, *Commun. Math. Phys.* **103**, 669 (1986).

[20] R.E. Peierls, *Proc. Roy. Soc.* **A214**, 143 (1952).

[21] R.M. Wald *Chicago Lectures in Physics: Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (Univ. of Chicago, Chicago, 1994).